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Superintegrability, Lax matrices and separation of variables [1]

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Abstract

We show how the superintegrability of certain systems can be deduced from the presence of multiple parameters in the rational Lax matrix representation. This is also related to the fact that such systems admit a separation of variables in parametric families of coordinate systems.

1 Rational Lax matrix representations of integrable systems

1.1 Classical R-matrix theory of commuting isospectral flows

In the classical R-matrix approach to finite dimensional integrable systems [7, 2, 4], there is a Poisson map from the phase space into a space of $r \times r$ Lax matrices $\mathcal{N}(\lambda)$ depending rationally, trigonometrically or elliptically on a spectral parameter λ. The Poisson bracket is defined by the relation

$$\{\mathcal{N}(\lambda) \overset{\otimes}{,} \mathcal{N}(\mu)\} = [r(\lambda-\mu),\ \mathcal{N}(\lambda)\otimes\mathbf{I} + \mathbf{I}\otimes\mathcal{N}(\mu)]\ , \tag{1}$$

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where both sides are interpreted as elements of $\mathrm{End}(\mathbf{C}^r \otimes \mathbf{C}^r)$. The symbol $\{\ \overset{\otimes}{,}\ \}$ signifies a simultaneous tensor product in $\mathrm{End}(\mathbf{C}^r) \sim \mathbf{gl}(r)$ and Poisson brackets in the components, and $r(\lambda - \mu)$ denotes the classical $R-$matrix. The simplest case is the rational R-matrix,

$$r(\lambda) := \frac{P_{12}}{\lambda}, \qquad P_{12}(\mathbf{u} \otimes \mathbf{v}) := \mathbf{v} \otimes \mathbf{u} \ , \tag{2}$$

with $\mathcal{N}(\lambda)$ a rational function of λ.

$$\mathcal{N}(\lambda) \ = \ \mathcal{B}(\lambda) + \sum_{i=1}^{n} \sum_{a=1}^{n_i} \frac{N_{ia}}{(\lambda - \alpha_i)^i} \tag{3}$$

$$\mathcal{B}(\lambda) \ := \ \sum_{i=1}^{n_0} B_i \lambda^i \ , \qquad\qquad N_{ia}, B_i \in \mathbf{gl}(r) \ . \tag{4}$$

Equations (1), (2) define the standard linear, rational R-matrix structure.

It follows from the properties of classical R-matrices [2, 7] that elements of the algebra of spectral invariants $\phi(\mathcal{N}) \in \mathcal{I}(\widetilde{\mathbf{gl}}(r))$ Poisson commute amongst themselves and generate commuting isospectral flows determined by the Lax equations:

$$\dot{\mathcal{N}} = \pm[(\mathbf{d}\phi_{\mathcal{N}})_{\pm}, \mathcal{N}] \ , \tag{5}$$

where $\mathcal{N}$ is here thought of as an element of the loop algebra $\widetilde{\mathbf{gl}}(r)$, identified in a standard way with its dual $\widetilde{\mathbf{gl}}^*(r)$ through the trace-residue pairing, and $(\ .\ )_{\pm}$ denotes projection to the $\pm$ components relative to the usual splitting of the loop algebra into positive and negative components

$$\widetilde{\mathbf{gl}}(r)) = \widetilde{\mathbf{gl}}(r))_{+} + \widetilde{\mathbf{gl}}(r))_{-} \tag{6}$$

(i.e. those admitting holomorphic continuations to the interior (+) and exterior (-) of the unit circle respectively with the latter normalized to vanish at ∞). The spectral invariants generate a maximal Poisson commuting algebra on generic symplectic leaves, defining completely integrable systems [3, 4]; i.e., there are as many functionally independent generators as half the dimension of the leaf.

1.2 2×2 rational Lax matrices

In the following, we shall limit our discussion to the case of 2×2 Lax matrices, although most of the considerations that follow are easily extended to higher rank. We may without loss of generality take $\mathcal{N}(\lambda)$ to be traceless (since the trace coefficients are Casimirs)

$$\mathcal{N}(\lambda) = \begin{pmatrix} h(\lambda) & e(\lambda) \\ f(\lambda) & -h(\lambda) \end{pmatrix} \ , \tag{7}$$

where the rational functions $e(\lambda), f(\lambda), h(\lambda)$ satisfy the Poisson bracket relations

$$\begin{aligned}
\{h(\lambda),\ e(\mu)\} &= \frac{e(\lambda)-e(\mu)}{\lambda-\mu} \\
\{h(\lambda),\ f(\mu)\} &= -\frac{f(\lambda)-f(\mu)}{\lambda-\mu} \\
\{e(\lambda),\ f(\mu)\} &= -2\frac{h(\lambda)-h(\mu)}{\lambda-\mu}\ .
\end{aligned} \tag{8}$$

For this case, the ring $\mathcal{I}(\widetilde{\mathbf{gl}}(2))$ of spectral invariants, when restricted to the symplectic leaves of the R-matrix Poisson structure, is generated by the quadratic trace invariants; i.e., the coefficients determining the numerator of the rational function

$$\Delta(\lambda) := -\frac{1}{2}\mathrm{tr}(\mathcal{N}^2(\lambda)) = h^2(\lambda) - \frac{1}{2}\left(e(\lambda)f(\lambda) + f(\lambda)e(\lambda)\right)\ . \tag{9}$$

(The order in the last two terms is irrelevant of course, but it is written here in a form that will also be valid in the quantum version below.) If, for example, the polynomial part $\mathcal{B}(\lambda)$ of $\mathcal{N}(\lambda)$ is taken to vanish, and only first order poles appear in $\mathcal{N}(\lambda)$, we have

$$\begin{aligned}
e(\lambda) &:= \sum_{i=1}^{n}\frac{e_i}{\lambda-\alpha_i} \\
f(\lambda) &:= \sum_{i=1}^{n}\frac{f_i}{\lambda-\alpha_i} \\
h(\lambda) &:= \sum_{i=1}^{n}\frac{h_i}{\lambda-\alpha_i},
\end{aligned} \tag{10}$$

where the quantities $\{e_i, f_i, h_i\}_{i=1\ldots n}$ are a set of n $\mathbf{sl}(2)$ generators, which may be canonically coordinatized as:

$$\begin{aligned}
e_i &:= \frac{1}{2}\left(y_i^2 + \frac{\mu_i^2}{x_i^2}\right) \\
f_i &:= \frac{1}{2}x_i^2 \\
h_i &:= \frac{1}{2}x_i y_i, \qquad i = 1, \ldots n,
\end{aligned} \tag{11}$$

where $\{\mu_i^2\}_{i=1\ldots n}$ are the values of the $\mathbf{sl}(2)$ Casimir invariants and $\{x_i, y_i\}_{i=1\ldots n}$ form a set of canonical coordinates on the symplectic leaves .

1.2.1 Parametric dependence of invariants and superintegrability

Again, taking the case when the polynomial part $\mathcal{B}(\lambda)$ of $\mathcal{N}(\lambda)$ vanishes (but not necessarily just first order poles), a complete set of generators is given by

$$\phi_{ia} := \mathrm{res}_{\lambda=\alpha_i}(\lambda-\alpha_i)^a \mathrm{tr}(\mathcal{N}^2(\lambda)), \qquad i = 1,\ldots n,\ a = 0,\ldots n_i - 1\ . \tag{12}$$

These commute amongst themselves, but they each depend upon the pole locations $\{\alpha_i\}_{i=1\ldots n}$ in $\mathcal{N}(\lambda)$. However, the linear combination:

$$\phi_{SI} := \sum_{i=1}^{n} \alpha_i \phi_{i0} = \mathrm{res}_{\lambda=\infty} \mathrm{tr}(\mathcal{N}^2(\lambda)) \tag{13}$$

does not depend on the α_i's. In general, there is no reason for the invariants $\phi_{ia}(\alpha_i)$ to commute with each other for different choices of the α_i's. But, regardless of the values chosen, they will commute with ϕ_{SI}. Since the $\phi_{ia}(\alpha_i)$'s for different choices of α_i's in general do not generate the same algebra of functions, we may conclude that, taken together, for different evaluations of the parameters $\{\alpha_i\}$, there are more functionally independent integrals that Poisson commute with ϕ_{SI} than half the dimension of the symplectic leaf, and hence the Hamiltonian system it generates is superintegrable. (In fact, in most cases, it may be shown to be maximally superintegrable; see the examples below.)

In particular, if we take the case of purely simple poles as above in (10), the resulting Hamiltonian is:

$$\phi_{SI} = \frac{1}{2}\sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2 - \frac{1}{2}(\sum_{i=1}^{n} x_i y_i)^2 + \frac{1}{2}\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} \frac{\mu_i^2}{x_i^2}\ , \tag{14}$$

which, when constrained to the (co)tangent bundle of the $n-1$ sphere S^{n-1}

$$\sum_{i=1}^{n} x_i^2 = 1, \qquad \sum_{i=1}^{n} x_i y_i = 0\ , \tag{15}$$

yields the superintegrable system

$$h_{\mathrm{Ros}} = \frac{1}{2}\sum_{j=1}^{n} y_j^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{\mu_i^2}{x_i^2}\ , \tag{16}$$

which is the trivial case of the Rosochatius system (without a harmonic oscillator potential).

1.2.2 Separation of variables

Another viewpoint that helps to explain the superintegrability of systems arising in this way is to note that they may be completely separated in a canonical coordinate system determined by the values of the pole parameters $\{\alpha_i\}$ which, for the $\mathbf{sl}(2)$ case with simple poles with the phase space constrained to S^{n-1} as above reduces to the sphero-conical system $\{\lambda_i, \zeta_i\}_{i=1\ldots n-1}$ defined by:

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} = \frac{\prod_{j=1}^{n-1}(\lambda - \lambda_j)}{\prod_{i=1}^{n}(\lambda - \alpha_i)}, \qquad \zeta_i := \frac{1}{2}\sum_{i=1}^{n} \frac{x_i y_i}{(\lambda - \alpha_i)} \ . \tag{17}$$

These are just the points (λ_i, ζ_i) on the invariant spectral curve

$$\zeta^2 + \frac{1}{2}\Delta(\lambda) = 0 \tag{18}$$

where the matrix element $f(\lambda)$ vanishes and $\zeta_i = h(\lambda_i)$ are the eigenvalues at these points. These are particular cases of the spectral *Darboux coordinates* of [3, 4]. (Note that these become hyperellipsoidal coordinates if there is a constant term added in the definition (10) of $f(\lambda)$.)

The point to note is that the separation of variables occurs in these coordinates simultaneously for *all* the invariants ϕ_{ia}, viewed as generators of Hamiltonian flows. But again, since the leading term spectral invariant ϕ_{SI} does not depend on the values of the parameters α_i, it admits a separation of variables in *any* of the family of sphero-conical (or hyperellipsoidal) coordinates obtained by varying these parameters. This simultaneous separability in multiple coordinates may be viewed as an alternative explanation of the origin of the superintegrability of such systems. (In fact, both these viewpoints are a result of the classical r-matrix setting, and in a sense may be considered as equivalent.)

In the examples given below in the following section, the same principle is used to deduce superintegrable systems from $\mathbf{sl}(2)$ Lax matrices satisfying the Poisson bracket relations (1).

1.2.3 Quantum integrable systems

The above discussion is easily extended to the canonically quantized version of such systems. All that must be done is to replace the matrix elements defining $\mathcal{N}(\lambda)$ by their quantized forms $\hat{e}(\lambda), \hat{f}(\lambda), \hat{h}(\lambda)$, which must satisfy the commutator analogs of the Poisson bracket relations (8)

$$[\hat{h}(\lambda),\ \hat{e}(\mu)] \ = \ \frac{\hat{e}(\lambda) - \hat{e}(\mu)}{\lambda - \mu}$$

$$
\begin{aligned}
[\hat{h}(\lambda),\ \hat{f}(\mu)] &= -\frac{\hat{f}(\lambda)-\hat{f}(\mu)}{\lambda-\mu} \\
[\hat{e}(\lambda),\ \hat{f}(\mu)] &= -2\frac{\hat{h}(\lambda)-\hat{h}(\mu)}{\lambda-\mu}, \ .
\end{aligned}
\tag{19}
$$

These can be realized by canonical quantization of the underlying classical phase space variables. For example, in the case of simple poles only, with vanishing polynomial term $\mathcal{B}(\lambda)$, we have:

$$
\begin{aligned}
\hat{e}(\lambda) &:= \sum_{i=1}^{n} \frac{\hat{e}_i}{\lambda-\alpha_i} \\
\hat{f}(\lambda) &:= \sum_{i=1}^{n} \frac{\hat{f}_i}{\lambda-\alpha_i} \\
\hat{h}(\lambda) &:= \sum_{i=1}^{n} \frac{\hat{h}_i}{\lambda-\alpha_i},
\end{aligned}
\tag{20}
$$

where the $\mathbf{sl}(2)$ generators $\{\hat{e}_i, \hat{f}_i, \hat{h}_i\}$ may be represented by the operators

$$
\begin{aligned}
\hat{e}_i &:= \frac{1}{2}\left(\frac{\partial^2}{\partial x_i^2} - \frac{\mu_i^2}{x_i^2}\right) \\
\hat{f}_i &:= \frac{1}{2}x_i^2 \\
\hat{h}_i &:= \frac{1}{2}\left(x_i\frac{\partial}{\partial x_i} + \frac{1}{2}\right), \qquad i = 1, \dots n,
\end{aligned}
\tag{21}
$$

and the commuting invariants are similarly given by the coefficients of the numerator polynomial of the quantum spectral invariant:

$$
\hat{\Delta}(\lambda) := \hat{h}^2(\lambda) - \frac{1}{2}\Big(\hat{e}(\lambda)\hat{f}(\lambda) + \hat{f}(\lambda)\hat{e}(\lambda)\Big) \ . \tag{22}
$$

The resulting systems are similarly quantum integrable, and separable in the same coordinates as the classical ones [6] and, for the same reasons as above, the quantum version of the Hamiltonian ϕ_{SI} is superintegrable.

In the following section, a number of examples of such classical and quantum superintegrable systems will be given.

2 Examples of superintegrable classical and quantum systems

The examples given below arise in the framework of the so-called Krall-Scheffer problem [8] of describing all two-dimensional analogs of classical orthogonal polynomials which

result in nine classes of second-order partial differential equations on the plane or on constant curvature surfaces. It was shown in [5],[9] that all nine cases are connected with superintegrable systems. The following are some illustrative examples.

2.1 Example 1. The sphere.

2.1.1 Classical Lax Matrix

The first case corresponds to three simple poles and vanishing $B(\lambda)$. The Lax matrix has the form:

$$N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\beta)} + \frac{N_3}{(\lambda-\gamma)} = \begin{pmatrix} h(\lambda) & f(\lambda) \\ e(\lambda) & -h(\lambda) \end{pmatrix} \tag{23}$$

where the matrix elements of the N_i generate a Poisson bracket realization of $(\mathbf{sl}(2))^3$:

$$N_1 = \frac{1}{2}\begin{pmatrix} s_1 p_1 & {p_1}^2 + \frac{\mu_1^2}{s_1^2} \\ -{s_1}^2 & -s_1 p_1 \end{pmatrix} \tag{24}$$

$$N_2 = \frac{1}{2}\begin{pmatrix} s_2 p_2 & {p_2}^2 + \frac{\mu_2^2}{s_2^2} \\ -{s_2}^2 & -s_2 p_2 \end{pmatrix} \tag{25}$$

$$N_3 = \frac{1}{2}\begin{pmatrix} s_3 p_3 & {p_3}^2 + \frac{\mu_3^2}{s_3^2} \\ -{s_3}^2 & -s_3 p_3 \end{pmatrix} \tag{26}$$

Here (p_1, p_2, p_3) are canonically conjugate to (s_1, s_2, s_3) (and these coincide with the coordinates $\{x_i, y_i\}_{i=1\ldots n}$ above).

2.1.2 Commuting invariants

The invariants are the coefficients of:

$$-\frac{1}{2}\mathrm{tr}N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\beta)} + \frac{H_3}{(\lambda-\gamma)} + \frac{{\mu_1}^2}{(\lambda-\alpha)^2} + \frac{{\mu_2}^2}{(\lambda-\beta)^2} + \frac{{\mu_3}^2}{(\lambda-\gamma)^2} \ . \tag{27}$$

Note that only two of the integrals H_1, H_2 and H_3 are independent, since their sum is zero. The Hamiltonian of the problem is given by their linear combination:

$$H = \alpha H_1 + \beta H_2 + \gamma H_3 = \frac{1}{2}({p_1}^2 + {p_2}^2 + {p_3}^2) + \frac{{\mu_1}^2}{{s_1}^2} + \frac{{\mu_2}^2}{{s_2}^2} + \frac{{\mu_3}^2}{{s_3}^2}. \tag{28}$$

This describes the Rosochatius system with harmonic oscillator terms absent on the cotangent bundle of a two-sphere in $\mathbb{R}^3$:

$$ {s_1}^2 + {s_2}^2 + {s_3}^2 = 1, \qquad s_1 p_1 + s_2 p_2 + s_3 p_3 = 0 \ . \tag{29} $$

The integrals H_1, H_2 and H_3 are as follows:

$$
\begin{aligned}
H_1 &= -\frac{1}{2}\frac{{L_{13}}^2 + {\mu_3}^2{s_1}^2/{s_3}^2 + {\mu_1}^2{s_3}^2/{s_1}^2}{\alpha-\gamma} - \frac{1}{2}\frac{{L_{12}}^2 + {\mu_1}^2{s_2}^2/{s_1}^2 + {\mu_2}^2{s_1}^2/{s_2}^2}{\alpha-\beta} \\
H_2 &= -\frac{1}{2}\frac{{L_{23}}^2 + {\mu_3}^2{s_2}^2/{s_3}^2 + {\mu_2}^2{s_3}^2/{s_2}^2}{\beta-\gamma} + \frac{1}{2}\frac{{L_{12}}^2 + {\mu_1}^2{s_2}^2/{s_1}^2 + {\mu_2}^2{s_1}^2/{s_2}^2}{\alpha-\beta} \\
H_3 &= \frac{1}{2}\frac{{L_{23}}^2 + {\mu_3}^2{s_2}^2/{s_3}^2 + {\mu_2}^2{s_3}^2/{s_2}^2}{\beta-\gamma} + \frac{1}{2}\frac{{L_{13}}^2 + {\mu_3}^2{s_1}^2/{s_3}^2 + {\mu_1}^2{s_3}^2/{s_1}^2}{\alpha-\gamma}
\end{aligned} \ , \tag{30}
$$

where $L_{ij} = s_1 p_2 - s_2 p_1$.

Note that the Hamiltonian H is independent of the parameters (α, β, γ), whereas the invariants H_1, H_2 do depend on them. Therefore, different choices for the parameters give distinct integrals that commute with H, but do not commute with each other.

2.1.3 Separating coordinates

The separating coordinates (λ_1, λ_2) in this case are sphero-conical coordinates. The corresponding momenta are denoted (ξ_1, ξ_2) . They are related to (s_1, s_2, s_3) and (p_1, p_2, p_3) by:

$$ {s_1}^2 = \frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)(\alpha-\gamma)} \qquad \xi_1 = -\frac{1}{2}\left(\frac{s_1p_1}{\lambda_1-\alpha} + \frac{s_2p_2}{\lambda_1-\beta} + \frac{-s_1p_1 - s_2p_2}{\lambda_1-\gamma}\right) \tag{31} $$

$$ {s_2}^2 = \frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\beta-\alpha)(\beta-\gamma)} \qquad \xi_2 = -\frac{1}{2}\left(\frac{s_1p_1}{\lambda_2-\alpha} + \frac{s_2p_2}{\lambda_2-\beta} + \frac{-s_1p_1 - s_2p_2}{\lambda_2-\gamma}\right) \tag{32} $$

$$ {s_3}^2 = \frac{(\gamma-\lambda_1)(\gamma-\lambda_2)}{(\gamma-\alpha)(\gamma-\beta)} \tag{33} $$

2.1.4 Quantum system

The quantum versions of the integrals above, denoted $\hat{H}_1, \hat{H}_2, \hat{H}_3$, are obtained by replacing the matrix elements of $N(\lambda)$ by the corresponding differential operators, $\hat{e}(\lambda), \hat{f}(\lambda), \hat{h}(\lambda)$, which in the case of simple poles are as in (20)-(22).

The quantization procedure leads to replacing the L_{ij}'s by their quantum version:

$$ \hat{L}_{ij} = \sqrt{-1}(s_i\partial/\partial s_j - s_j\partial/\partial s_i) \tag{34} $$

Introducing the functions

$$\omega_{jk}^2 := {\mu_j}^2 {s_k}^2/{s_j}^2 + {\mu_k}^2 {s_j}^2/{s_k}^2 \qquad j,k = 1..3 \tag{35}$$

and denoting $\alpha = \alpha_1$, $\beta = \alpha_2$, $\gamma = \alpha_3$, we can present the quantum integrals as

$$\hat{H}_i = -\frac{1}{2}\sum_{k\neq i} \frac{\hat{L}_{ik} + \omega_{ik}^2}{\alpha_i - \alpha_k} \qquad i,k = 1..3 \tag{36}$$

The quantum Hamiltonian is

$$\hat{H} = -\frac{1}{2}({\partial_1}^2 + {\partial_2}^2 + {\partial_3}^2) + \frac{{\mu_1}^2}{{s_1}^2} + \frac{{\mu_2}^2}{{s_2}^2} + \frac{{\mu_3}^2}{{s_3}^2}. \tag{37}$$

The separating coordinates are the configuration space part of the ones for the classical case (λ_1, λ_2).

2.2 Example 2. The hyperboloid.

2.2.1 Classical Lax Matrix

Consider now a Lax matrix with one first order and one second order pole:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \alpha)^2} + \frac{N_3}{(\lambda - \beta)}, \tag{38}$$

with

$$\begin{aligned} N_1 &= \frac{1}{2}\begin{pmatrix} s_1 p_1 + s_2 p_2 & 2p_1 p_2 + 2\gamma_1\gamma_2 \\ -2s_1 s_2 & -s_1 p_1 - s_2 p_2 \end{pmatrix} \\ N_2 &= \frac{1}{2}\begin{pmatrix} -s_2 p_1 & -{p_1}^2 - \gamma_2^2 \\ {s_2}^2 & s_2 p_1 \end{pmatrix} \\ N_3 &= \frac{1}{2}\begin{pmatrix} s_3 p_3 & {p_3}^2 + \gamma_3^2 \\ -{s_3}^2 & -s_3 p_3 \end{pmatrix} \end{aligned} \tag{39}$$

Here we have introduced the following notations

$$2\gamma_1\gamma_2 := \frac{2\mu_2^2 s_1}{s_2^3} - \frac{2\mu_1\mu_2}{s_2^2} \ , \ \gamma_2^2 := -\frac{{\mu_2}^2}{s_2^2} \ , \ \gamma_3^2 := \frac{\mu_3^2}{s_3^2}. \tag{40}$$

The matrix elements of (N_1, N_2) generate a Poisson bracket realization of the jet extension $\mathbf{sl}(2)^{(1)*}$ while those of N_3 generate a second $\mathbf{sl}(2)$.

2.2.2 Commuting invariants

The trace formula again gives us only two independent commuting invariants H_1 and H_2

$$-\frac{1}{2}\mathrm{tr}N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\alpha)^2} - \frac{\mu_1\mu_2}{(\lambda-\alpha)^3} + \frac{{\mu_2}^2}{2(\lambda-\alpha)^4} + \frac{H_3}{(\lambda-\beta)} - \frac{{\mu_3}^2}{2(\lambda-\beta)^2} \quad (41)$$

since, by taking the residue we obtain

$$H_1 + H_3 = 0 \ . \quad (42)$$

The superintegrable Hamiltonian in this case is:

$$H = (\alpha-\beta)H_1 + H_2 - \frac{1}{2}{\mu_3}^2 = 2p_1p_2 - {p_1}^2 + {p_3}^2 + 2\gamma_1\gamma_2 - \gamma_2^2 + \gamma_3^2. \quad (43)$$

The quadratic constraint now defines a hyperboloid

$$2s_1s_2 + {s_3}^2 = 1 \ . \quad (44)$$

In the ambient coordinates the integrals H_1 and H_2 are

$$\begin{aligned} H_1 &= \frac{(s_1p_3 - s_3p_2)(s_3p_1 - s_2p_3) - \gamma_3^2 s_1s_2 - 2\gamma_1\gamma_2 s_3^2}{\alpha-\beta} - \frac{((s_3p_1 - s_2p_3)^2 + \gamma_3^2 s_2^2 + \gamma_2^2 s_3^2}{2(\alpha-\beta)^2} \\ H_2 &= \frac{1}{2}(s_1p_1 - s_2p_2)^2 - 2\gamma_1\gamma_2 s_1 + \frac{(s_3p_1 - s_2p_3)^2 + \gamma_3^2 s_2^2 + \gamma_2^2 s_3^2}{2(\alpha-\beta)}. \end{aligned} \quad (45)$$

Again, whereas the Hamiltonian H does not depend on the parameters (α, β) the integrals $H_1,\ H_2$ do, which provides an explanation for the superintegrability in this case.

2.2.3 Separating coordinates

These are determined by the relations:

$${s_3}^2 = \frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)^2} \qquad \xi_1 = -\frac{1}{2}\left(\frac{s_1p_1 + s_2p_2}{\lambda_1-\alpha} - \frac{s_2p_1}{(\lambda_1-\alpha)^2} + \frac{s_3p_3}{\lambda_1-\beta}\right) \quad (46)$$

$${s_2}^2 = -\frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)} \qquad \xi_2 = -\frac{1}{2}\left(\frac{s_1p_1 + s_2p_2}{\lambda_2-\alpha} - \frac{s_2p_1}{(\lambda_2-\alpha)^2} + \frac{s_3p_3}{\lambda_2-\beta}\right) \quad (47)$$

$$s_1s_2 = -\frac{1}{2}\left(\frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)^2} - 1\right). \quad (48)$$

2.2.4 Quantum system

The quantized integrals $\hat{H}_1, \hat{H}_2, \hat{H}_3$ are obtained as before by replacing all conjugate variables by the corresponding differential operators. The quantum integrals may then be expressed as

$$\begin{aligned}
\hat{H}_1 &= -\frac{(s_1\partial_3 - s_3\partial_2)(s_3\partial_1 - s_2\partial_3) + \gamma_3^2 s_1 s_2 + 2\gamma_1\gamma_2 s_3^2}{\alpha-\beta} + \frac{(s_3\partial_1 - s_2\partial_3)^2 - \gamma_3^2 s_2^2 - \gamma_2^2 s_3^2}{2(\alpha-\beta)^2} \\
\hat{H}_2 &= \frac{1}{2}\hat{L}_{12}^2 - 2\gamma_1\gamma_2 s_1 - \frac{(s_3\partial_{s_1} - s_2\partial_{s_3})^2 - \gamma_3^2 s_2^2 - \gamma_2^2 s_3^2}{2(\alpha-\beta)},
\end{aligned} \tag{49}$$

where $\partial_k := \partial/\partial s_k$.

The quantum Hamiltonian is

$$\hat{H} = 2\partial_1\partial_2 - {\partial_1}^2 + {\partial_3}^2 + 2\gamma_1\gamma_2 - \gamma_2^2 + \gamma_3^2, \tag{50}$$

and this again separates in the configuration space coordinates (λ_1, λ_2).

2.3 Example 3. The plane.

2.3.1 Classical Lax Matrix

For the cases with zero curvature like the example to follow, the polynomial part $\mathcal{B}(\lambda)$ of the Lax matrix does not vanish. The simplest case involves two distinct finite poles in $N(\lambda)$ and constant $B(\lambda)$

$$\begin{aligned}
N(\lambda) = \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix} + \frac{1}{2(\lambda-\alpha)} \begin{pmatrix} s_1 p_1 & {p_1}^2 + \frac{\mu_1^2}{s_1^2} \\ -{s_1}^2 & -s_1 p_1 \end{pmatrix} \\
+ \frac{1}{2(\lambda-\beta)} \begin{pmatrix} s_2 p_2 & {p_2}^2 + \frac{\mu_2^2}{s_2^2} \\ -{s_2}^2 & -s_2 p_2 \end{pmatrix} .
\end{aligned} \tag{51}$$

The matrix elements of the residues $N_1,\ N_2$ generate two copies of $\mathbf{sl}(2)$.

2.3.2 Commuting invariants

The invariants of motion are defined by:

$$-\frac{1}{2}\mathrm{tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)} + \frac{H_2}{(\lambda-\beta)} + \frac{{\mu_1}^2}{(\lambda-\alpha)^2} + \frac{{\mu_2}^2}{(\lambda-\beta)^2} - a. \tag{52}$$

The superintegrable Hamiltonian in this case is given by

$$H = \frac{1}{4}\mathrm{res}_\infty \mathrm{tr} N(\lambda)^2 = \frac{1}{4}({p_1}^2 + {p_2}^2 + a({s_1}^2 + {s_2}^2) + \frac{{\mu_1}^2}{{s_1}^2} + \frac{{\mu_2}^2}{{s_2}^2}), \tag{53}$$

which gives an isotropic oscillator together with Rosochatius terms. As before (p_1, p_2) are canonically conjugate to (s_1, s_2).

In terms of the ambient space coordinates the integrals H_1 and H_2 are :

$$\begin{aligned} H_1 &= p_1^2 + as_1^2 + \frac{\mu_1^2}{s_1^2} - \frac{1}{2(\alpha-\beta)}(L_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}) \\ H_2 &= p_2^2 + as_2^2 - \frac{\mu_2^2}{s_2^2} + \frac{1}{2(\alpha-\beta)}(L_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}). \end{aligned} \tag{54}$$

Where $L_{12} := s_1 p_2 - s_2 p_1$ and $H = \frac{1}{4}(H_1 + H_2)$. Here the additional integral results from the parametric dependence on $(\alpha - \beta)$.

2.3.3 Separating coordinates

The separating coordinates $(\lambda_1, \lambda_2, \xi_1, \xi_2)$ in this case are defined by

$$ {s_1}^2 = 2\frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)} \qquad \xi_1 = -\frac{1}{2}(\frac{s_1 p_1}{\lambda_1 - \alpha} + \frac{s_2 p_2}{\lambda_1 - \beta}) \tag{55}$$

$$ {s_2}^2 = -2\frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\alpha-\beta)} \qquad \xi_2 = -\frac{1}{2}(\frac{s_1 p_1}{\lambda_2 - \alpha} + \frac{s_2 p_2}{\lambda_2 - \beta}) \tag{56}$$

2.3.4 Quantum system

The Hamiltonian of the corresponding quantum problem is

$$\hat{H} = \frac{1}{4}\mathrm{res}_\infty \mathrm{tr}\hat{N}(\lambda)^2 = \frac{1}{4}({\partial_1}^2 + {\partial_2}^2 + a({s_1}^2 + {s_2}^2) + \frac{{\mu_1}^2}{{s_1}^2} + \frac{{\mu_2}^2}{{s_2}^2}) = \frac{1}{4}(\hat{H}_1 + \hat{H}_2). \tag{57}$$

The quantum integrals $\hat{H}_1$ and $\hat{H}_2$ are:

$$\begin{aligned} \hat{H}_1 &= \partial_1^2 + as_1^2 + \frac{\mu_1^2}{s_1^2} - \frac{1}{2(\alpha-\beta)}(\hat{L}_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}) \\ \hat{H}_2 &= \partial_2^2 + as_2^2 - \frac{\mu_2^2}{s_2^2} + \frac{1}{2(\alpha-\beta)}(\hat{L}_{12}^2 + \frac{\mu_1^2 s_2^2}{s_1^2} + \frac{\mu_2^2 s_1^2}{s_2^2}) \end{aligned} \tag{58}$$

and the separating coordinates are again (λ_1, λ_2), which depend on the additional parameter $(\alpha - \beta)$.

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